Synchronizing the information content of a chaotic map and flow via symbolic dynamics

Ned J. Corron* and Shawn D. Pethel
U. S. Army Aviation and Missile Command, AMSAM-RD-WS-ST, Redstone Arsenal, Alabama 35898

Krishna Myneni
Science Applications International Corporation, 6725 Odyssey Drive, Huntsville, Alabama 35806
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In this paper we report an extension to the concept of generalized synchronization for coupling different types of chaotic systems, including maps and flows. This broader viewpoint takes disparate systems to be synchronized if their information content is equivalent. We use symbolic dynamics to quantize the information produced by each system and compare the symbol sequences to establish synchronization. A general architecture is presented for drive-response coupling that detects symbols produced by a chaotic drive oscillator and encodes them in a response system using the methods of chaos control. We include experimental results demonstrating synchronization of information content in an electronic oscillator circuit driven by a logistic map.

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Chaotic oscillators have positive Shannon entropy and can be viewed as information sources [1]. Previously, it has been demonstrated that an information signal can be encoded in the symbolic dynamics of a chaotic oscillator [2,3]. This encoding can be performed using methods of chaos control, whereby a natural unstable trajectory of the oscillator can be targeted using small control perturbations [4,5]. Consequently, the methods of chaos control may be used to take information exhibited by one chaotic oscillator and encode it in the dynamics of a second oscillator. If the two oscillators are identical, the result is a synchronized state, where both oscillators exhibit the same wave form [6]. In fact, the quality of synchronization can be made arbitrarily high provided the synchronization channel capacity is equal to or greater than the Shannon entropy [7]. For dissimilar oscillators, the wave forms are not the same, yet the information content may still be synchronized.

To compare the information generated by different chaotic oscillators, it is expedient to use symbolic dynamics to describe the state-space trajectory for each system. In the theory of symbolic dynamics, a trajectory is represented by a sequence of symbols corresponding to a coarse-grained partitioning of the system's state space [8]. For a *generating partition* each possible trajectory corresponds to a distinct symbol sequence; conversely, each observed symbol sequence uniquely identifies a system trajectory. Since identical oscillators share the same symbolic dynamics, synchronization of two or more identical oscillators requires each to generate the same symbol sequence relative to a common generating partition. Such oscillators are effectively producing the same information, which is neatly quantized and labeled by the symbolic representation. Viewing synchronization as an information-matching process, we can extend synchronization to mismatched or entirely dissimilar oscillators. We recognize that two arbitrary oscillators are perfectly synchronized in an information sense if they produce the same information—i.e., symbols generated by one system map injectively to symbols emitted by the other. True synchronization requires that the common information be emitted at precisely the same time; however, for mismatched oscillators this requirement is too strict, especially when considering flows where the system return time can vary depending on the precise trajectory. A more practical requirement is that information is emitted at the same average rate or entropy. For unidirectional coupling, a significant delay between the drive and response systems may be allowed for certain applications.

A general architecture for implementing drive-response synchronization using a symbol channel is shown in Fig. 1. At the drive system, a free-running chaotic oscillator is generating information, which is captured in a symbolic representation using a symbol detector. The symbols are transmitted to the response system, where a controller converts the symbols into precise control signals used to guide the response oscillator to the desired trajectory. Also shown in the figure is a common phase reference, which indicates that some external drive is required to assure that symbols are generated at the same average rate in the two oscillators.

Previously, the concept of *generalized synchronization* was introduced to investigate coupling of mismatched or dissimilar systems [9,10]. Customarily, it is said that coupled systems exhibit generalized synchronization if their states are related by an invertible function. This definition of generalized synchronization must be contained within the information view of synchronization, since an invertible function necessarily preserves the information content of a signal. However, the synchronization of symbolic information is broader and can describe synchronization of more diverse

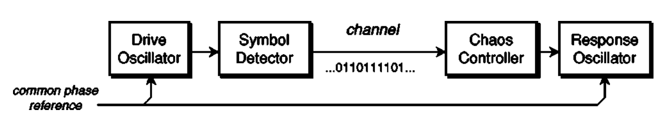

FIG. 1. General architecture for drive-response synchronization using a symbol channel.

*Electronic address: ned.corron@ws.redstone.army.mil

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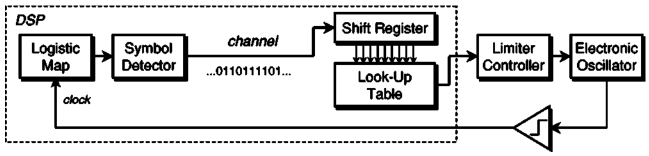

FIG. 2. Experimental configuration to couple a logistic map and an electronic oscillator for drive-response synchronization via a symbol channel.

systems. For example, coupling iterated maps to continuous flows is not accommodated by prior definitions of generalized synchronization.

The measure of *mutual information* is an effective tool for detecting redundancy in sequences and flows [11]. It has been used effectively to quantify the degree of synchronization in coupled flows [12,13]. The mutual information between two oscillators can be significant even when the systems are dissimilar, such as a discrete map coupled to a continuous flow. Here we exploit symbolic dynamics to quantize the information and use chaos control to ensure the symbols are identical in the two systems. As a result, the mutual information in the symbol sequences produced by each oscillator is maximal, and we recognize the oscillators to be perfectly synchronized in an information sense.

To illustrate synchronization of symbolic information, we provide an experimental demonstration of drive-response synchronization using a discrete map and an electronic oscillator. The drive system is a chaotic logistic map, which evolves at discrete time steps, while the response system is a chaotic electronic oscillator that evolves continuously in time. The experimental configuration is shown in Fig. 2. In this experiment, the logistic map, its symbol detector, the channel, as well as part of the chaos controller are implemented in software using a DSP card (Innovative Integration ADC64) hosted in a PC. The response oscillator, a limiter controller, and a phase detector are all implemented in hardware using discrete electronic components. The phase detector provides a clock signal to the DSP for iterating the logistic map; thus, the map is effectively phase locked to the external oscillator and guaranteed to generate symbols at the natural return rate of the oscillator.

For the drive system, we use the logistic map defined by the iteration

$$x_{i+1}=\mu x_i(1-x_i), \tag{1}$$

where i is an index for the iterates and $0\leqslant\mu\leqslant 4$ is a system parameter that is adjusted to obtain chaos. For one-dimensional, unimodal maps a generating partition exists at the maximum and its dynamics are completely represented using two symbols. Thus, for each iterate of the map we generate a symbol s_i according to

$$s_i=\begin{cases}0, & x_i<x^*,\\ 1, & x_i\geqslant x^*,\end{cases} \tag{2}$$

where $x^*=0.5$ is the generating partition. These symbols are transmitted through the channel, where they are accumulated in an N-bit shift register. With each received symbol, the register is shifted one bit toward the most significant bit, which is discarded, and the new symbol enters as the least significant bit. The shift register then provides an index into an N-bit look-up table, from which a control level is selected and output via a digital-to-analog converter. For all experiments shown here we use $N=12$, which is consistent with the precision of the digital-to-analog interface available on the DSP card.

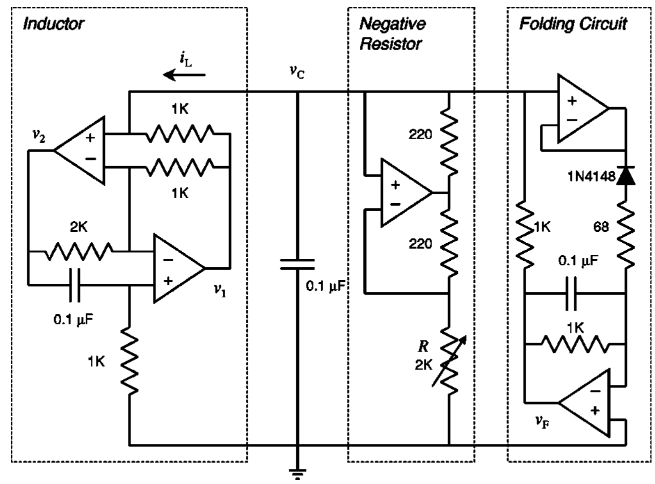

FIG. 3. Chaotic electronic oscillator used as the response system synchronized symbolically to a logistic map.

The chaotic electronic oscillator is shown in Fig. 3. All operational amplifers are type AD712, all resistors are 1% tolerance, and the circuit is supplied with ± 15 V. This circuit uses an unstable LC harmonic oscillator coupled to a nonlinear folding circuit to generate chaos. The inductor is implemented using a general impedance converter, from which the inductor current i_L is derived using

$$i_L=\frac{v_2-v_1}{3\ \mathrm{k}\Omega}. \tag{3}$$

With $R\simeq 800\ \Omega$, the uncontrolled circuit oscillates near 1 kHz and exhibits a simply folded band attractor similar to that of Rossler's oscillator [14]. In Fig. 4, the return map of the peaks in the tank voltage v_C shows that the oscillator dynamics are well modeled by a one-dimensional, unimodal return map, which we represent as

$$\xi_{i+1}=f(\xi_i), \tag{4}$$

where $\xi_i=v_C(t_i)$ is the ith positive peak in the tank voltage for the response oscillator. For each peak ξ_i, we define a symbol σ_i as

$$\sigma_i=\begin{cases}0, & \xi_i<\xi^*,\\ 1, & \xi_i\geqslant\xi^*,\end{cases} \tag{5}$$

where $\xi^*=0.310$ V approximates the generating partition at the maximum of the return map in Fig. 4.

This circuit is readily controlled using a limiter applied to the tank voltage [15]. Here, we use an extension of limiter control called dynamic limiting, which enables practical chaos control of arbitrary wave forms [16]. In dynamic lim-

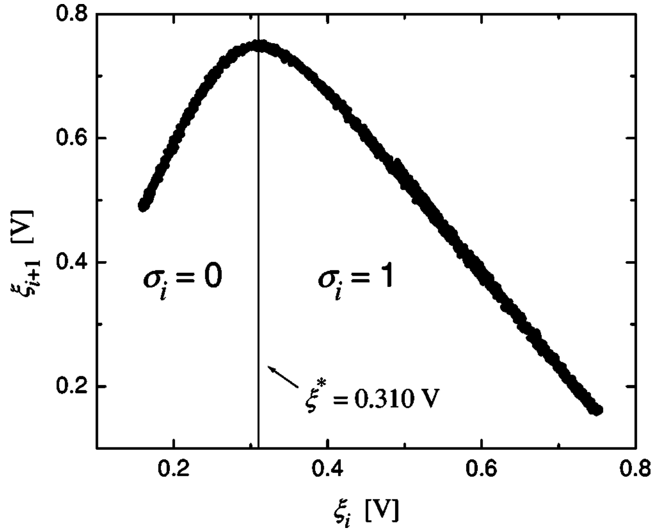

FIG. 4. Observed peak return map for the uncontrolled electronic oscillator displaying one-dimensional, unimodal dynamics. The vertical line at $\xi^* = 0.310$ V approximates a generating partition for the response system.

iting, the tank voltage is limited to a prescribed level that is set to a new value for each peak return. In the circuit, we use an active limiter documented previously [16]. The limiter levels are selected from the N-bit look-up table stored in the DSP using the shift register as the index. The look-up table contains control points (or, in effect, initial conditions) for the circuit to generate the various N-bit symbol sequences. As such, applying a limiter level from the table controls the circuit to exhibit the same symbolic information as the logistic map. By inserting the new symbols into the least significant position in the shift register, we are assured that the

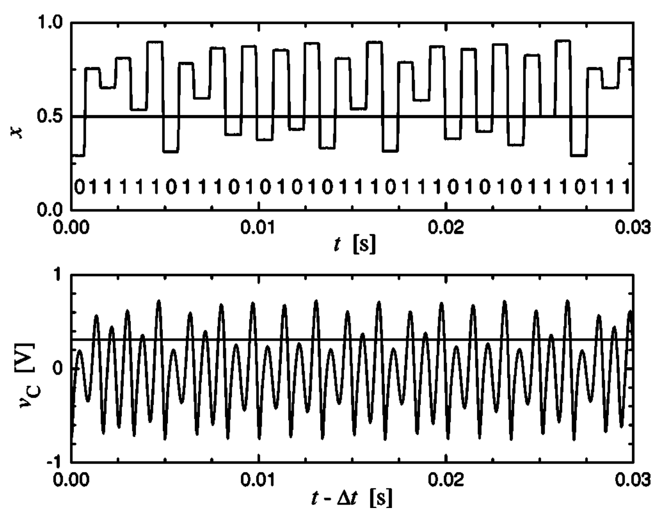

FIG. 5. Output states of the driving logistic map (top) and synchronized electronic circuit (bottom) for the map parameter $\mu = 3.64$. The generating partition for each system is shown as the horizontal line, and the bottom plot is shifted by the delay $\Delta t = 0.092$ s due to the shift register. Both systems exhibit the same symbol sequence, which is shown just above the time axis in the top plot.

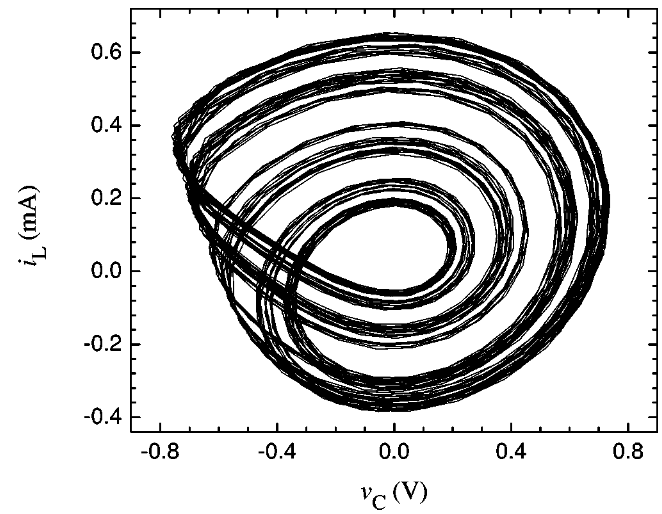

FIG. 6. Observed attractor for the electronic circuit synchronized to the logistic map with parameter $\mu = 3.64$.

desired response is elicited from the circuit using only small perturbations. The data for the look-up table is collected by observing the uncontrolled dynamics of the circuit and binning the peak voltages together with the corresponding N-bit symbol sequence. For each observed sequence, an entry in the look-up table is then taken to be the mean of all peak values associated with that sequence. For $N = 12$, we use 200 000 peak returns to generate the look-up table for this circuit.

To synchronize the information content of the two oscillators, it is necessary that the response system have sufficient information capacity to encode any possible symbol sequence generated by the drive system. In other words, any arbitrary symbol sequence generated by the drive must uniquely map to an admissible sequence in the response system. For one-dimensional, unimodal maps, kneading theory provides a simple requirement to guarantee that the sequences produced by the drive are directly admissible in the

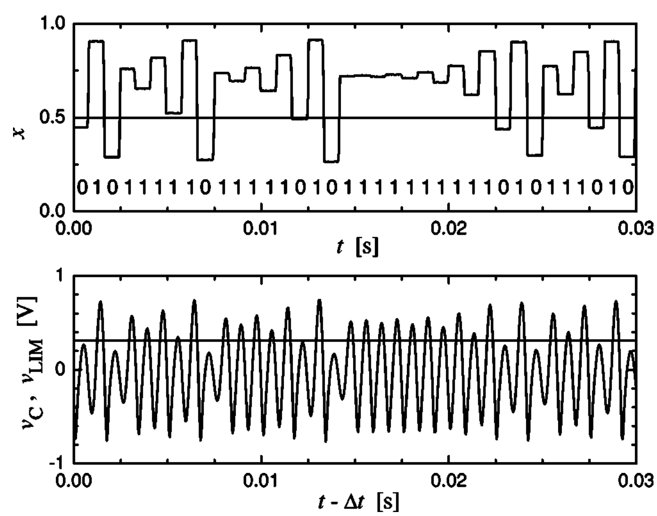

FIG. 7. Output states of the driving logistic map (top) and synchronized electronic circuit (bottom) and the common symbol sequence for the map parameter $\mu = 3.68$.

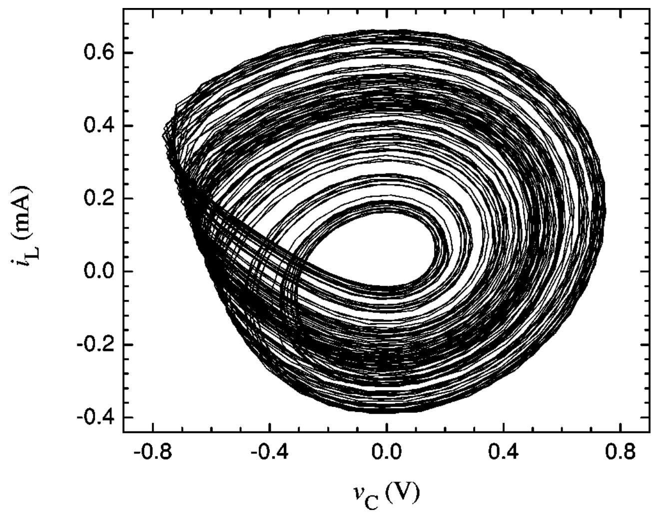

FIG. 8. Observed attractor for the electronic circuit synchronized to the logistic map with parameter $\mu = 3.68$.

response [17]. Applied here, the requirement is satisfied if the topological entropy of the response system matches or exceeds that of the drive system.

In our experimental configuration, we can directly adjust the topological entropy of both the drive and response systems. The parameter μ controls the entropy exhibited by the logistic map; loosely, the entropy increases with μ, attaining an unrestricted grammar of two symbols at $\mu = 4$. In the circuit, the variable resistor R controls the entropy of the response system. In the experiment, we determined that for $\mu < \mu_c \simeq 3.7$ the sequences generated by the logistic map are compatible with the electronic circuit and can be controlled using small perturbations. However, for $\mu > \mu_c$, the sequences are not generally admissible, which is manifest by intermittent failure of the chaos control algorithm to encode the response system dynamics via small perturbations.

Example output for $\mu = 3.64$ is shown in Fig. 5. In the top trace, iterates of the driving logistic map are plotted as a function of continuous time. In addition, the partition at $x^* = 0.5$ is shown, and a symbol is defined for each iterate relative to this partition. In the bottom plot, the tank voltage of the controlled circuit is shown. Here, the time scale is shifted by $\Delta t = 0.092$ s to account for the delay due to the 12-bit shift register. The partition for the peak returns is shown at $\xi^* = 0.310$ V. Thus, the information content of the drive and response systems can be compared in these plots. For each iterate of the return map above or below the partition in the top plot, there is a corresponding peak in the tank voltage v_C also above or below the partition, indicating a high level of mutual information. The corresponding attractor for the controlled circuit is shown in Fig. 6. For this value of the map parameter, the entropy of the response system greatly exceeds that of the drive, and the Cantor-like structure of the attractor indicates that only a small portion of the response system's grammar is being used.

Example output for $\mu = 3.68$ is shown in Fig. 7, and the corresponding controlled attractor is shown in Fig. 8. Here we note that the topological entropy of the drive system is roughly equivalent to that of the response oscillator. Consequently, the grammar of the response system is more fully used. This is apparent in the attractor shown in Fig. 8, where the controlled system exhibits nearly the full attractor for the response system. Making μ much larger results in a failure of the synchronization process, since bit sequences generated by the drive are inconsistent with the natural dynamics of the response system. In this case, the control points are undefined and the response system cannot be controlled to track the information content of the drive system.

In this experiment, we note that the synchronization of information is achieved at the cost of a delay between the drive and response systems. That is, the information exhibited by the response system lags the drive by a time Δt, which is proportional to N, the depth of the look-up table. Practically, the reconstruction of the wave form based on detecting the current symbol of the drive and controlling future symbols in the response will lead to a delay; however, such delay can be avoided by detecting future symbols in the current system state [18].

In summary, we have shown an extension to the concept of generalized synchronization by considering the symbolic information content of two coupled systems. This view of synchronization requires that synchronized systems exhibit equivalent information at the same average rate. With this extension, one can now consider coupling dissimilar systems, including a continuous flow coupled to a discrete map.

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